Symplectic aspects of the tt*-Toda equations and the constant problem

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The tt*-Toda equations

- The equations are appeared in the work of Cecotti and Vafa on two-dimensional supersymmetric field theories.
- Some solutions correspond to the quantum products of weighted projective spaces.

Definition

Let $n \in \mathbb{Z}_{>0}$.

$$2\left(w_i\right)_{z\bar{z}} = -e^{2(w_{i+1}-w_i)} + e^{2(w_i-w_{i-1})},$$

$$w_i : \mathbb{C}^* \to \mathbb{R},\ i \in \mathbb{Z},\ w_i = w_{i+n+1},\ w_i = w_i(|z|),$$

$$\begin{cases} w_0 + w_{-1} = 0, & w_1 + w_{-2} = 0,\ \ldots \\ w_0 + w_n = 0, & w_1 + w_{n-1} = 0,\ \ldots \end{cases}.$$

Isomonodromy and Riemann-Hilbert correspondence

The tt*-Toda equations are equivalent to the isomonodromic condition of the following meromorphic connection on $\mathbb{C}P^1$:

$$\frac{d\Psi}{d\zeta} = \left(-\frac{1}{\zeta^2} W - \frac{1}{\zeta} x w_x + x^2 W^T \right) \Psi,$$

$$w = \mathrm{diag}(w_0, \ldots, w_n), \ W = \begin{pmatrix} & e^{w_1 - w_0} & & \\ & & \ddots & \\ & & & e^{w_n - w_{n-1}} \\ e^{w_0 - w_n} & & & \end{pmatrix}$$

Hamiltonian structure

Proposition (O.)

The tt-Toda equations are equivalent to the following Hamiltonian system:* $(x = |z|)$

$$H(w_0, \ldots, w_{\frac{n-1}{2}}, \tilde{w}_0, \ldots, \tilde{w}_{\frac{n-1}{2}}; x)$$

$$:= \frac{1}{2x} \sum_{i=0}^{\frac{n-1}{2}} \tilde{w}_i^2 - x \sum_{i=1}^{\frac{n-1}{2}} e^{2(w_i - w_{i-1})} - \frac{x}{2} \left(e^{-4w_{\frac{n-1}{2}}} + e^{4w_0} \right)$$

$$(w_i)_x = \frac{\partial H}{\partial \tilde{w}_i} = \frac{\tilde{w}_i}{x}$$

$$(\tilde{w}_i)_x = -\frac{\partial H}{\partial w_i} = -2x \left(e^{2(w_{i+1} - w_i)} - e^{2(w_i - w_{i-1})} \right)$$

Correspondence between asymptotic data and monodromy data

Let

$$2w_i(x) \sim \gamma_i \log x + \rho_i + o(1),\ (x \to 0,\ x = |z|),$$

be asymptotic expansions of the solutions of the tt*-Toda equations in the generic case. It holds that

$$m_i = -\frac{1}{2}\gamma_i,\ e_i^{\mathbb{R}} = e^{\rho_i} 2^{2\gamma_i} \frac{X_{n-i}}{X_i}$$

$$X_i(\gamma_0, \dots, \gamma_n) := \prod_{j=1}^{n} \Gamma\left(\frac{\gamma_{i+j} - \gamma_i + 2j}{2(n+1)}\right).\ (\gamma_{j+n+1} = \gamma_j)$$

Correspondence between asymptotic data and monodromy data

parametrizing local solutions

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Correspondence between asymptotic data and monodromy data

parametrizing global solutions

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Generating function 1

Theorem (O.)

The Riemann-Hilbert correspondence preseves natural symplectic structures, i.e.,

$$\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} d\gamma_i \wedge d\rho_i = -2 \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} dm_i \wedge d \log e_i^{\mathbb{R}}.$$

The generating function of the above canonical transformation is:

$$F(\rho_0, \ldots, \rho_{\lfloor \frac{n-1}{2} \rfloor}, m_0, \ldots, m_{\lfloor \frac{n-1}{2} \rfloor}) := - \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \rho_i m_i + 2 \log 2 \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} m_i^2$$

$$+ \frac{n+1}{2} \sum_{k=0}^{n} \sum_{j=1}^{n} \psi^{(-2)} \left(\frac{m_{k-j} - m_k + j}{n+1} \right). \ (m_{j+n+1} = m_j = -m_{n-j})$$

Generating function 2

In general,

$$m_i = -\frac{\partial F}{\partial \rho_i}, \ \log e_i^{\mathbb{R}} = -\frac{\partial F}{\partial m_i}.$$

If the solutions w_i are globally smooth, it holds that $\log e_i^{\mathbb{R}} = 0$, so

$$\rho_i = -(2\log 2)\gamma_i + \log(X_i/X_{n-i}).$$

Noting that $\frac{\partial}{\partial m_i} = -2\frac{\partial}{\partial \gamma_i}$, it holds that

$$\frac{\partial}{\partial \gamma_i}\left(F + \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \rho_i m_i\right) = \rho_i.$$

Asymptotic constant of the tau-function

Definition

Define a function τ by

$$\log \tau(x) = \int_1^x H(w_i(s), \tilde{w}_i(s), s) ds.$$

Define constants C_0, C_∞ depending on $\gamma_0, \ldots, \gamma_{\lfloor (n-1)/2 \rfloor}$ by:

$$\tau(x) = C_0 x^{\frac{1}{8} \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \gamma_i^2} (1 + o(1)), \; x \to 0$$

$$\tau(x) = C_\infty e^{-x^2} (1 + o(1)), \; x \to \infty$$

Asymptotic constant of the tau-function

$$\log C_0 = \lim_{x\to 0}\left(\log \tau(x) - \frac{1}{8}\sum_{i=0}^{\lfloor (n-1)/2\rfloor} \gamma_i^2 \log x\right)$$

$$\log C_\infty = \lim_{x\to\infty}\left(\log \tau(x) + x^2\right)$$

$$C := \log \frac{C_\infty}{C_0} = \lim_{\substack{x_1\to 0\\ x_2\to\infty}}\left(\log \frac{\tau(x_2)}{\tau(x_1)} + x_2^2 + \frac{1}{8}\sum_{i=0}^{\left\lfloor \frac{n-1}{2}\right\rfloor} \gamma_i^2 \log x_1\right)$$

$$= \lim_{\substack{x_1\to 0\\ x_2\to\infty}}\int_{x_1}^{x_2}\left(H(w(x),\tilde{w}(x),x) + x_2^2 + \frac{1}{8}\sum_{i=0}^{\left\lfloor \frac{n-1}{2}\right\rfloor} \gamma_i^2 \log x_1\right)dx$$

Constant problem

Theorem (O.)

It holds that

$$C = -\sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \frac{\gamma_i^2}{8} - \frac{1}{2}\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \gamma_k \rho_k - \frac{1}{4}F + \frac{1}{4}\, F|_{\gamma=0}\,.$$

Example

If $n = 3$, $\frac{1}{4}\, F|_{\gamma=0} = 2\left(\psi^{(-2)}(\frac{1}{4}) + \psi^{(-2)}(\frac{2}{4}) + \psi^{(-2)}(\frac{3}{4})\right).$

Sketch of the proof 1

- The Hamiltonian function H is quasi-homogeneous:

$$H(w, c\tilde{w}; cx) = cH(w, \tilde{w}; x) \text{ for any } c > 0.$$

- Such a Hamiltonian function satisfies some identity, and in this case:

$$H = \tilde{w}_0 \left(w_0\right)_x + \cdots + \tilde{w}_{\lfloor \frac{n-1}{2} \rfloor} \left(w_{\lfloor \frac{n-1}{2} \rfloor}\right)_x - H + \frac{d}{dx}(xH).$$

- The classical action $S(x_1, x_2) := \int_{x_1}^{x_2} \left(\sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \tilde{w}_i \left(w_i\right)_x - H\right) dx$ appears:

$$\log \frac{\tau(x_2)}{\tau(x_1)} = S(x_1, x_2) + xH(x)|_{x_1}^{x_2}$$

Sketch of the proof 2

$$\frac{\partial S(x_1, x_2)}{\partial \gamma_j} = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \left(\tilde{w}_k \left(w_k \right)_{\gamma_j} \right) \Big|_{x_1}^{x_2}$$

$$\frac{\partial C}{\partial \gamma_i} = -\frac{\gamma_i}{4} - \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{\gamma_k}{4} \left(\rho_k \right)_{\gamma_i}$$

$$C = -\sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \frac{\gamma_i^2}{8} - \frac{1}{4} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \gamma_k \rho_k + \frac{1}{4} \int \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \rho_k d\gamma_k$$

$$\int \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \rho_k d\gamma_k = -F - \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \gamma_k \rho_k + \text{const.}$$

Remarks

Remark

The symplectic form $\sum_{i=0}^{\lfloor (n-1)/2 \rfloor} dm_i \wedge d \log e_i^{\mathbb{R}}$ can be considered as the Atiyah-Hitchin form on the space $R = \{p/q | \deg p \leq n, \deg q = n+1, q : \text{monic}\}$ of the based rational maps from $\mathbb{C}P^1$ to itself of degree $n+1$.

Remark

The symplectic form $\sum_{i=0}^{\lfloor (n-1)/2 \rfloor} d\gamma_i \wedge d\rho_i$ can be considered as the Kirillov-Kostant form of a coadjoint orbit of a subgroup of $GL_{n+1}\left(\mathbb{C}[\zeta]/\left(\zeta^2\right)\right)$.

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